F. anal. 17.2. 2025

- Ingen slider, info ?

Chp. 5: The F.-transform in $\mathbb{R}$

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \xi}\, dx$$

A. Integration on $\mathbb{R}$

Improper R.-integral:

$$\int_{-\infty}^{\infty} f(x)\, dx \overset{\text{def.}}{=} \lim_{N\to\infty} \int_{-N}^{N} f(x)\, dx$$

$\int_{-N}^{N}$ ex. if f R.-int. or cont. on $[-N, N]$,

$\lim_{N\to\infty}$ ex. if f has sufficient decay at ∞.

Moderate decrease: $\exists\ \varepsilon > 0$ and $A > 0$ s.t.

(1) $$|f(x)| \leq \frac{A}{(1+|x|^2)^{\frac{1+\varepsilon}{2}}},\ x \in \mathbb{R} \quad \left(\leq \frac{A}{|x|^{1+\varepsilon}} \right)$$

Cont. f'ns of moderate decrease:

$$C_m(\mathbb{R}) := \{ f : f \in C(\mathbb{R}),\ f \text{ satisfy } (1) \}$$

$$(= M(\mathbb{R}) \text{ in SS})$$

Rem. 1: $C_m(\mathbb{R})$ is a vector sp.

Lem. 1: $f \in C_m(\mathbb{R}) \Rightarrow \int_{-\infty}^{\infty} f(x)\, dx$ (and $\int_{-\infty}^{\infty} |f(x)|\, dx$) exists

Pf.:

$$I_N \doteq \int_{-N}^{N} f(x)\,dx \quad \text{Cauchy sequence in } \mathbb{R}:$$

$$\underset{N<M}{|I_N - I_M|} \leq \int_{N\leq|x|\leq M} |f(x)|\,dx \overset{(1)}{\leq} \int_{N\leq|x|\leq M} \frac{A}{|x|^{1+\varepsilon}}\,dx$$

$$= 2A\int_N^M \frac{1}{x^{1+\varepsilon}}\,dx = \frac{2A}{-\varepsilon}\left(\frac{1}{M^\varepsilon} - \frac{1}{N^\varepsilon}\right) \leq \frac{2A}{\varepsilon N^\varepsilon} \underset{N\to\infty}{\longrightarrow} 0$$

$$\Longrightarrow \{I_N\}_N \text{ Cauchy in } \mathbb{R} \overset{\mathbb{R}\text{ complete}}{\Longrightarrow} I = \lim_{N\to\infty} I_N \text{ ex.} \quad \square$$

Lem. 2: Properties of $\int_{-\infty}^{\infty}$

$f, g \in C_m(\mathbb{R})$, $a, b \in \mathbb{C}$, $\delta > 0$, $h \in \mathbb{R}$

(a) $\int_{-\infty}^{\infty}(af + bg) = a\int_{-\infty}^{\infty} f + b\int_{-\infty}^{\infty} g$ (lin.)

(b) $\int_{-\infty}^{\infty} f(x-h)\,dx = \int_{-\infty}^{\infty} f(x)\,dx$ (translation inv.)

(c) $\delta\int_{-\infty}^{\infty} f(\delta x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx$ (scaling prop.)

(d) $\int_{-\infty}^{\infty} |f(x-h) - f(x)|\,dx \underset{h\to 0}{\longrightarrow} 0.$ (cont. of transl.)

Pf:

(b) $\left|\int_{-\infty}^{\infty} f(x-h) - \int_{-\infty}^{\infty} f(x)\right|$

$$\leq \underbrace{\int_{|x|>N} |f(x-h)|}_{\overset{(1)}{=} O(N^{-\varepsilon})} + \underbrace{\left|\int_{|x|<N} f(x-h) - \int_{|x|<N} f(x)\right|}_{} + \underbrace{\int_{|x|>N} |f(x)|}_{\overset{(1)}{=} O(N^{-\varepsilon})}$$

$$\overset{y=x-h}{=} \left|\int_{-N-h}^{N-h} f(y)\,dy - \int_{-N}^{N} f(x)\,dx\right|$$

$$= \left(\int_{-N-h}^{-N} + \int_{N-h}^{N}\right)|f(x)|\,dx \overset{(1)}{=} O(N^{-\varepsilon})$$

$$\overset{N\to\infty}{\longrightarrow} 0$$

(c) Similar to (b) (Exercise 7)

(d) Let $\varepsilon > 0$. Take $|h| < 1$.

$$\int_{-\infty}^{\infty} |f(x-h) - f(x)|\,dx \le \underbrace{\int_{|x|>N} (|f(x-h)| + |f(x)|)}_{\overset{(1)}{=} O(N^{-\varepsilon})} + \underbrace{\int_{|x|<N} |f(x-h) - f(x)|}_{\le \sup_x(\text{---}) = \omega_{f,[-N,N]}(h)}$$

$\uparrow$ f unif. cont. on $[-N,N]$

$$\le \underbrace{C \cdot N^{-\varepsilon}}_{< \frac{\varepsilon}{2} \text{ for } N \text{ large}} + \underbrace{2N \cdot \omega_{f,[-N,N]}(h)}_{< \frac{\varepsilon}{2} \text{ for } h \text{ small s.t. } 2N\omega_f(h) < \frac{\varepsilon}{2}} < \varepsilon$$

□

Rem. 2:

(a) $\int_{-\infty}^{\infty}$ well-def. and Lem. 2 holds for f in:

(i) $PC_m(\mathbb{R})$ = p.w. cont. f'ns of moderate decrease

(ii) $R_m(\mathbb{R})$ = R-int. f'ns on $[-N,N]$ $\forall N \in \mathbb{N}$, moderate decrease

(iii) $L^1(\mathbb{R})$, but now $\int$ is the Lebesgue int.

Pf Lem. 2: (a)–(c) unchanged, (d) approx. by cont. f'ns

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(b) From Lebesgue theory: $p \in [1,\infty)$

(2) $f \in L^p(\mathbb{R}) \Rightarrow \int_{|x|>N} |f|^p \xrightarrow[N\to\infty]{} 0$ (dom. conv. thm.)

(3) $f \in L^p(\mathbb{R}) \Rightarrow \exists f_n \in C(\mathbb{R})$ s.t. $\|f - f_n\|_p \xrightarrow[n\to\infty]{} 0$

$(f_n \in C_c(\mathbb{R}))$ comp. supp.

$\|f\|_p = (\int_{\mathbb{R}} |f|^p)^{\frac{1}{p}}$

B. Def. of F-transf.

Def. 1: F.-transform of $f \in C_m(\mathbb{R})$ is

(4) $$\mathcal{F}[f](\xi) = \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \xi}\, dx, \quad \forall\, \xi \in \mathbb{R}$$

Rem. 3: $f \in C_m(\mathbb{R})$

(a) $\hat{f} : \mathbb{R} \to \mathbb{C}$ is bnd., and

$$\|\hat{f}\|_\infty \underset{|e^{-2\pi i x \xi}| = 1}{\leq} \int_{-\infty}^{\infty} |f(x)|\, dx = \|f\|_1$$

(b) $\hat{f} \in C_0(\mathbb{R})$, cont. and $|\hat{f}(\xi)| \underset{|\xi| \to 0}{\longrightarrow} 0$ [Exercise 7]

(c) Problem: $\hat{f} \notin C_m$ or L^1 in general ...

How to make sense of $\mathcal{F}^{-1}[\hat{f}]\ (= f)$?

Rem. 4: $f \in PC_m(\mathbb{R})$ (or $L^1(\mathbb{R})$)

$\Rightarrow$ (4) is well-def. and Rem. 3 still holds.

C. The Schwartz space $\mathcal{S}(\mathbb{R})$

Nice space where $\mathcal{F}, \mathcal{F}^{-1}$ makes sence

Def. 2:

(a) $f \in C^\infty(\mathbb{R})$ is <u>rapidly decreasing</u> if

$$(5) \qquad \sup_{x \in \mathbb{R}} |x|^k |f^{(l)}(x)| < \infty \qquad \forall\ k, l \in \mathbb{N}$$

(b) <u>Schwartz space</u>

$$S(\mathbb{R}) := \{ f \in C^\infty(\mathbb{R}) : f \text{ satisfy } (5) \}$$

Ex. 1: $f(x) = e^{-ax^2} \in S(\mathbb{R})$ if $a > 0$

Chk: $f^{(l)}(x) = P_l(x) e^{-ax^2}$, P_l l-order poly.

$$\Rightarrow |x|^k |f^{(l)}(x)| \leq C(1 + |x|^{k+l}) e^{-ax^2} \xrightarrow[|x| \to \infty]{} 0$$

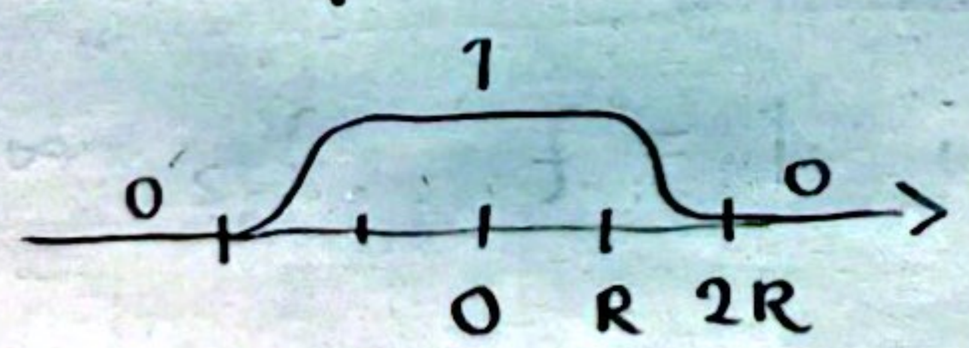
$\Rightarrow$ (5) holds.

Ex. 2: $f_1(x) = \dfrac{1}{1+x^2}$, $f_2(x) = e^{-|x|} \notin S(\mathbb{R})$

$f_1 \in C^\infty$, but $|x|^3 |f(x)| \xrightarrow[|x| \to \infty]{} \infty$ (unbd.)

f_2 decays fast, but $f_2'(x) = \frac{x}{|x|} e^{-|x|}$ not cont. (at $x = 0$)

Ex. 3: Bump-f'n

$f_R =$ 

$0 \leq f_R \in C^\infty$, $f_R = 1, |x| < R$, $f_R = 0, |x| > 2R$

$\Rightarrow f_R \in S(\mathbb{R})$ $\quad [\, f_R^{(l)}(x) = 0,\ |x| > 2R,\ \forall\, l \in \mathbb{N} \,]$

$f_R \in C_c^\infty(\mathbb{R}) := \{ f \in C^\infty(\mathbb{R}) : \exists\, R > 0 \text{ s.t. } f(x) = 0 \text{ for } |x| > R \}$

"compact support"

$$C_c^\infty(\mathbb{R}) \subset S(\mathbb{R}) \subset C_m^\infty(\mathbb{R}) \subset C^\infty(\mathbb{R}).$$

moderate decrease

Ex. 4: Std. mollifier ("bump f'n" in SS)

$$\rho(x) = \begin{cases} e^{-\frac{1}{1-x^2}}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$

$$\rho \in C_c^\infty(\mathbb{R}) \subset S(\mathbb{R}) \quad [\text{Exercise } 7]$$

Obs: $\tilde{f}_R(x) = \chi_{[-R,R]} * \rho(x)$

satisfies Ex. 3 for $R > 1$.

F. anal. 20.2.2025

- slides, info

A. The F.-transf. on $S(\mathbb{R})$

$f \in S(\mathbb{R}) \subset C_m(\mathbb{R})$:

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx, \quad \forall \xi \in \mathbb{R}$$

"Known from prev. courses:"

Prop. 1: $f \in S(\mathbb{R})$, $h \in \mathbb{R}$, $\delta > 0$, $\xi \in \mathbb{R}$

(a) $\mathcal{F}[f(x+h)](\xi) = \hat{f}(\xi) e^{2\pi i h \xi}$ (transl. of f)

(b) $\mathcal{F}[f(x) \cdot e^{-2\pi i x h}](\xi) = \hat{f}(\xi + h)$ (transl. of $\hat{f}$)

(c) $\mathcal{F}[f(\delta x)](\xi) = \frac{1}{\delta} \hat{f}\left(\frac{\xi}{\delta}\right)$ (scaling)

(d) $\mathcal{F}[f'(x)](\xi) = 2\pi i \xi \hat{f}(\xi)$ $\left(\frac{d}{dx} f\right)$

(e) $\mathcal{F}[(-2\pi i x) f(x)](\xi) = \hat{f}'(\xi)$ $\left(\frac{d}{d\xi} \hat{f}\right)$

Rem 1:

(a) and (b): translation by h $\overset{\mathcal{F}/\mathcal{F}^{-1}}{\longleftrightarrow}$ mult. w. $e^{2\pi i h \xi} / e^{2\pi i h x}$

(d) and (e): derivative $\overset{\mathcal{F}/\mathcal{F}^{-1}}{\longleftrightarrow}$ mult. w. $2\pi i \xi / (-2\pi i x)$

Pf.:

(a) $\int_{-\infty}^{\infty} f(x+h) e^{-2\pi i x \xi} dx = e^{+2\pi i \xi h} \int_{-\infty}^{\infty} f(x+h) e^{-2\pi i (x+h)\xi} dx$

$\overset{\text{transl. inv.}}{\underset{\substack{\text{Lem. 2 b)}\\ \text{last time}}}{=}} e^{+2\pi i h \xi} \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$

(b) By def.

(c) $\int_{-\infty}^{\infty} f(\delta x) e^{-2\pi i x \xi} dx = \frac{1}{\delta} \cdot \delta \int_{-\infty}^{\infty} \underbrace{f(\delta x) e^{-2\pi i \delta x (\frac{\xi}{\delta})}}_{g(\delta x)} dx$

$\overset{\text{scaling prop.}}{\underset{\substack{\text{Lem. 2 c)}\\ \text{last time}}}{=}} \frac{1}{\delta} \int_{-\infty}^{\infty} g(x)\, dx = \frac{1}{\delta} \int_{-\infty}^{\infty} f(x) e^{-2\pi i x (\frac{\xi}{\delta})} dx$

$\underset{\text{def } \hat{f}}{=} \frac{1}{\delta} \hat{f}\left(\frac{\xi}{\delta}\right)$

(d) $\int_{-N}^{N} f'(x) e^{-2\pi i x \xi} dx \overset{\text{i.b.p.}}{=} \left[f(x) e^{-2\pi i x \xi} \right]_{x=-N}^{x=N}$

$+ 2\pi i \xi \int_{-N}^{N} f(x) e^{-2\pi i x \xi} dx$

$\downarrow$ $N \to \infty$, $f \in S(\mathbb{R})$

$\widehat{f'}(\xi) = 0 + 2\pi i \xi \hat{f}(\xi)$

(e) $\left| \frac{\hat{f}(\xi+h) - \hat{f}(\xi)}{h} - 2\pi i x \hat{f}(\xi) \right|$

$\overset{\text{def. } \hat{f}}{\underset{(b)}{=}} \left| \int_{-\infty}^{\infty} \left(\underbrace{\frac{e^{-2\pi i h x} - 1}{h}} - 2\pi i x \right) f(x) e^{-2\pi i x \xi} dx \right|$

$\overset{\text{fund. thm.}}{\underset{\text{of calc.}}{=}} \int_0^1 (-2\pi i x) e^{-2\pi i (th) x} dt$

$$\leq \int_{|x|\leq N} \Big| 2\pi i x \cdot \int_0^1 \underbrace{\left(e^{-2\pi i (th)x} - 1\right)}_{\overset{\text{f. thm calc.}}{=} \int_0^1 (-2\pi i t h x) e^{-2\pi i (sth)x} ds} dt \Big| \cdot |f(x)| \, dx$$

$$+ \int_{|x|>N} 2\pi |x| \Big(\int_0^1 \underbrace{\left|e^{-2\pi i (th)x}\right|}_{= 1} dx + 1 \Big) \cdot |f(x)| \, dx$$

$$\leq \int_{|x|\leq N} (2\pi)^2 h \underbrace{|x|^2 |f(x)|}_{\leq N^2 \|f\|_\infty} dx$$

$$+ \int_{|x|>N} 2\pi |x| \cdot 2 \cdot |f(x)| \, dx$$

$$\leq \underbrace{(2\pi)^2 h N^2 \|f\|_\infty \cdot 2N}_{< \frac{\varepsilon}{2}, \ h \text{ small (N fixed)}}$$

$$\underbrace{+ 4\pi \underbrace{\sup_{x\in\mathbb{R}} |x|^3 |f(x)|}_{< \infty, \ f \in S(\mathbb{R})} \underbrace{\int_{|x|>N} \frac{1}{|x|^2} dx}_{= O(N^{-1})}}_{< \frac{\varepsilon}{2} \ N \text{ large (indep. of } h)}$$

$$< \varepsilon$$

$\square$

Hence $\hat{f}'(\xi)$ exists and $= 2\pi i x \hat{f}(\xi)$ $\square$

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<u>Thm. 1:</u> $f \in S(\mathbb{R}) \Rightarrow \hat{f} \in S(\mathbb{R})$

<u>Pf.:</u>

$$\xi^k \hat{f}^{(l)}(\xi) \overset{\text{Prop. 1e, } l\text{-times}}{=} \xi^k \mathcal{F}\left[(-2\pi i x)^l f(x)\right](\xi)$$

$$\overset{\text{Prop. 1d, } k \text{ times}}{=} \frac{1}{(2\pi i)^k} \mathcal{F}\left[\left(\frac{d}{dx}\right)^k \left((-2\pi i x)^l f(x)\right)\right](\xi)$$

$$\sup_{\xi \in \mathbb{R}} |\xi|^k |\hat{f}^{(l)}(\xi)| \underset{\substack{\uparrow \\ |\hat{g}(\xi)| \le \|g\|_1}}{\le} \frac{1}{(2\pi)^k} \underbrace{\| \overbrace{(\tfrac{d}{dx})^k ((-2\pi i x)^l f(x))}^{g} \|_1}_{\le \underbrace{\|(1+x^2) g\|_\infty}_{<\infty \text{ since } g \in S(\mathbb{R})} \underbrace{\| \frac{1}{1+x^2} \|_1}_{<\infty}}$$

$$< \infty, \quad \forall\, l, k \in \mathbb{N}$$

$$\Rightarrow \hat{f} \in S(\mathbb{R}). \qquad \square$$

B. The Gaussian

Lem. 1: $\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$

Pf.: $\left(\int_{-\infty}^{\infty} e^{-\pi x^2} dx\right)^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)} dx\, dy$

$$= \int_0^{2\pi} \int_0^{\infty} e^{-\pi r^2} r\, dr\, d\theta$$

$$= 2\pi \int_0^{\infty} e^{-\pi r^2} r\, dr$$

$$\underset{du = 2\pi r\, dr}{\overset{u = \pi r^2}{=}} \left[-e^{-u}\right]_0^{\infty} = 1 \qquad \square$$

Thm 2.2: $f(x) = e^{-\pi x^2} \Rightarrow \hat{f}(\xi) = f(\xi) = e^{-\pi \xi^2}$

So: $\mathcal{F}[e^{-\pi x^2}] = e^{-\pi \xi^2}$, maps to itself!

Pf.: (Maf 4 ...)

1) $\hat{f}(0) = \int_{-\infty}^{\infty} e^{-\pi x^2} \cdot e^0\, dx \overset{\text{Lem. 1}}{=} 1$

2) $f'(x) \overset{\text{chk.}}{=} -2\pi x f(x)$

3) $\overset{\text{Prop. 1e}}{\Longrightarrow} \quad \hat{f}'(\xi) = \mathcal{F}[(-2\pi i x) f(x)](\xi)$

$$\overset{2)}{=} \mathcal{F}[\, i f'(x)](\xi)$$

$$\overset{\text{Prop. 1d}}{=} i \cdot 2\pi i \xi \hat{f}(\xi) = -2\pi \xi \hat{f}(\xi)$$

$$\overset{\text{separable ODE}}{\Longrightarrow} \hat{f}(\xi) = \hat{f}(0) e^{-\pi \xi^2} \overset{1)}{=} e^{-\pi \xi^2} \qquad \square$$

Gaussian:

$K_\delta(x) := \frac{1}{\sqrt{\delta}} e^{-\pi \frac{x^2}{\delta}}$, $\delta > 0$, $x \in \mathbb{R}$

Obs. 1: Scaling-invariance: $K_\delta(x) = \delta^{-\frac{1}{2}} K_1(\delta^{-\frac{1}{2}} x)$

Cor. 1: $\widehat{K_\delta}(\xi) = e^{-\pi \delta \xi^2}$

So: $\mathcal{F}[\frac{1}{\sqrt{\delta}} e^{-\pi \frac{x^2}{\delta}}](\xi) = e^{-\pi \delta \xi^2}$

Pf.: $\widehat{K_\delta}(\xi) \overset{\text{Obs. 1}}{=} \delta^{-\frac{1}{2}} \mathcal{F}[K_1(\delta^{-\frac{1}{2}} x)](\xi)$

$$\overset{\text{Prop. 1c}}{=} \delta^{-\frac{1}{2}} \left(\delta^{\frac{1}{2}} \widehat{K_1}(\delta^{\frac{1}{2}} \xi) \right)$$

$$\underset{\widehat{K_1} = e^{-\pi x^2}}{\overset{\text{Thm. 2}}{=}} e^{-\pi \delta \xi^2} \qquad \square$$

Thm. 3: $\{K_\delta\}_{\delta > 0}$ are good kernels/approx. δ's:

(i) $\int_{-\infty}^{\infty} K_\delta(x)\, dx = 1$

(ii) $\int_{-\infty}^{\infty} |K_\delta(x)|\, dx \le M \quad (M = 1)$

(iii) $\forall\, \eta > 0$, $\int_{|x| > \eta} |K_\delta(x)|\, dx \to 0$ as $\delta \to 0$

Pf.:

(i) $\int_{-\infty}^{\infty} K_\delta(x)\,dx = \widehat{K_\delta}(0) \overset{\text{Cor. 1}}{=} e^0 = 1$

(ii) $\int_{-\infty}^{\infty} |K_\delta| \overset{K_\delta \geq 0}{=} \int_{-\infty}^{\infty} K_\delta \overset{(i)}{=} 1$

(iii) $\int_{|x|>\eta} |K_\delta(x)| = \int_{|x|>\eta} e^{-\pi(\delta^{-\frac{1}{2}}x)^2}\,\delta^{-\frac{1}{2}}\,dx \underset{dy = \delta^{-\frac{1}{2}}dx}{\overset{y=\delta^{-\frac{1}{2}}x}{=}} \int_{|y|>\delta^{-\frac{1}{2}}\eta} \underbrace{e^{-\pi y^2}}_{\substack{\leq e^{-\pi|y|} \\ y>1}}\,dy$

$$\cancel{\leq \underbrace{e^{-\frac{1}{2}\pi(\delta^{-\frac{1}{2}}\eta)^2}}_{\substack{\downarrow\ \delta\to 0 \\ 0}} \underbrace{\int_{|y|>\delta^{-\frac{1}{2}}\eta} e^{-\frac{1}{2}\pi y^2}\,dy}_{\leq \int_{-\infty}^{\infty}(\,\cdots) \underset{dy=\sqrt{2}\,dz}{\overset{z=\frac{y}{\sqrt{2}}}{=}} \sqrt{2}\underbrace{\int_{-\infty}^{\infty} e^{-\pi z^2}\,dz}_{\overset{\text{Lem. 1}}{=} 1}}}$$

$$\overset{\delta^{-\frac{1}{2}}\eta>1}{\leq} 2\int_{\delta^{-\frac{1}{2}}\eta}^{\infty} e^{-\pi y}\,dy = \frac{2}{\pi}\left(e^{-\pi\delta^{-\frac{1}{2}}\eta} - 0\right) \underset{\delta\to 0}{\longrightarrow} 0$$

$\square$

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Rem. 2:

$f_\delta(x) = \delta f(\delta x) \Rightarrow \int_{-\infty}^{\infty} |f_\delta(x)|\,dx \overset{y=\delta x}{=} \int_{-\infty}^{\infty} |f(y)|\,dy \quad (f\in L^1, \delta>0)$